Journal of Global Optimization **7**: 261–277, 1995.

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Optimization over the Efficient Set

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(Received: 16 August 1994; accepted: 6 April 1995)

Abstract. This paper deals with the problem of maximizing a function over the efficient set of a linear multiple objective program. The approach is to formulate a biobjective program with an appropriate efficient set. The penalty function approach is motivated by an auxiliary problem due to Benson.

Key words: Optimization, efficiency set, linear multiple objective programming, penalty function.

1. Introduction and Preliminaries

The basis of multiple criteria decision making involves attempting to maximize several objectives simultaneously. Usually a single solution does not optimize all of the objectives simultaneously. This leads to a set of solutions which are commonly called efficient solutions. An outcome is considered to be desirable if no objective can be improved without a negative consequence to some other objective. The efficient set, E, is the set of alternatives with desirable outcomes.

In this work the objective functions are assumed to be linear. Much is known about E in this case including numerical methods for generating E when the set of alternatives is polyhedral. See [1, 2, 3, 4].

One problem that arises in applications is to determine the minimum value that each criterion may assume over the set of efficient alternatives. In this work the more general problem of maximizing a general function φ over E is considered. In practice it is desirable to solve this problem without generating E. Because E is rarely a convex set, φ may have local extrema which are not global. Moreover, the set where φ is optimized need not be connected.

Philip [3] considers the case where φ is linear and the set of alternatives, X, is polyhedral. He presents a simplex-like solution technique and a special case where it suffices to maximize φ over X. Benson [5, 6] extends the problem by allowing X to be a closed convex set. He analyzes several aspects of the problem and proves numerous results. Various authors [2, 7, 8, 9, 10] have considered the problem of minimizing the criterion functions over E. In particular, the case of minimizing a convex function over E when the constraint set is polyhedra has been studied by Bolintineanu [11].

Here the case where φ is an arbitrary continuous function and X is a compact convex set is investigated. See also Bolintineanu [12]. In Section 2 a special function originally defined by Benson [5] is presented with several of its important properties. Section 3 pairs this function with φ to form a biobjective problem whose efficient set compares favorably with the solutions to (P). In particular, a penalized problem [13] is obtained as a scalarization of the biobjective problem. A parametric programming technique for solving (P) is developed in this work. In Section 4 it is shown that the parametric program can be used to provide an approximate or exact solution to (P), depending upon the specific nature of X and φ. It is shown that if X is polyhedral and φ is convex, then the penalty is exact. Numerical aspects are discussed in Section 5.

Given a set S of feasible alternatives and a function $f : S \to R^k$, the efficient set of S with respect to f is defined as follows.

DEFINITION 1.1. A point $x \in S$ is said to be an *efficient point* of S with respect to f if there is no $s \in S$ such that $f(s) \geqslant f(x)$ and $f(s) \neq f(x)$. If such an $s \in S$ does exist, then $f(s)$ is said to *dominate* $f(x)$. The efficient set of S with respect to f is denoted by $E(S, f)$.

Let X be a nonempty, compact, convex subset of R^n. Let C be a k by n matrix with constant coefficients with rows c_i, $1 \leqslant i \leqslant k$. The following vector maximization problem corresponds to C and X

$$(V) \qquad V\text{MAX}\, Cx \quad \text{subject to} \quad x \in X.$$

The solutions of (V) are the elements of $E(X, C)$, which is nonempty.

For the central problem of this paper, let φ be a continuous real valued function on X and consider the problem

$$(P) \qquad \sup \varphi(x) \quad \text{subject to} \quad x \in E(X, C).$$

Let t^* denote the supremum of $\varphi(x)$ subject to $x \in E(X, C)$. Notice that t^* is finite [14]. If X is a polytope, then $E(X, C)$ is compact and, consequently, t^* is attained. In the nonpolyhedral case $E(X, C)$ is not necessarily closed and the optimal value need not be attained (see [15]).

The approach uses Benson's function $g(x)$ to construct the penalized problem

$$(P\lambda) \qquad \max(1, \lambda)(\varphi(x), -g(x))^T \quad \text{subject to} \quad x \in X.$$

If $g(X)$ is continuous, it is shown that the optimal value of $(P\lambda)$ tends to the optimal value of (P) as λ goes to infinity.

A special version of (P) considered in [2, 7, 8, 9, 10] is to solve, for each $1 \leqslant i \leqslant k$,

$$(P_i) \qquad \inf c_i x \quad \text{subject to} \quad x \in E(X, C).$$

In the following it is shown that solving (P_i) is equivalent to maximizing a particular convex function over X.

2. Benson's Measurement Function for Efficiency

In this section a function defined by Benson is examined. This function is known to indicate whether or not a point is efficient. Properties are presented which allow this function to be regarded as a measurement between a point and the efficient set.

In [5], Benson considers the function $g : G \to R$ given by

$$g(x) = \max e^T Cw - e^T Cx \quad \text{subject to} \quad Cw \geqslant Cx \text{ and } w \in X \tag{2.1}$$

where $G = \{x \in R^n : Cw \geqslant Cx \text{ for some } w \in X\}$ and $e = (1, 1, \ldots, 1)^T \in R^k$. Let $W(x) = \{w \in X : Cw \geqslant Cx\}$. Since X is a compact subset of G, the maximum in (2.1) is attained. The function is derived from a modification of a mathematical programming problem used to discuss the existence of efficient points (e.g., see [1]). The following propositions give properties of the function g [6, 14].

PROPOSITION 2.1. *For all $x \in G$, $g(x) \geqslant 0$, and for all $x \in X$ it follows that $g(x) = 0$ if and only if $x \in E(X, C)$. If $x \in G$ and $g(x) = 0$, then x is the sum of an efficient point and an element of the null space of C.*

PROPOSITION 2.2. *Let $x \in G$. If $w \in X$, $Cw \geqslant Cx$ and $g(x) = e^T C(w - x)$, then $w \in E(X, C)$. That is, for a fixed x the maximum in the mathematical program (2.1) can only be attained at an efficient point.*

PROPOSITION 2.3. *If X is a compact, convex set, then g is a concave, upper semicontinuous function on G. If X is a polytope, then g is a continuous piecewise linear function on G.*

COROLLARY 2.1. *The function g is continuous on $E(X, C)$ and on the interior of G. If g is continuous on G, then $E(X, C)$ is closed.*

PROPOSITION 2.4. *Let x, $\mathbf{x} \in G$.*

(a) If $Cx \geqslant C\mathbf{x}$, then $g(x) \leqslant g(\mathbf{x})$.

(b) If $Cx \geqslant C\mathbf{x}$ and $Cx \neq C\mathbf{x}$, then $g(x) < g(\mathbf{x})$.

Proof. Let $Cx \geqslant C\mathbf{x}$. Then $e^T Cx \geqslant e^T C\mathbf{x}$ and $W(x)$ is a subset of $W(\mathbf{x})$. Therefore

$$\begin{aligned} g(x) &= \max e^T C(w - x) && \text{subject to } w \in W(x) \\ &\leqslant \max e^T C(w - \mathbf{x}) && \text{subject to } w \in W(x) \\ &\leqslant \max e^T C(w - \mathbf{x}) && \text{subject to } w \in W(\mathbf{x}) \\ &= g(\mathbf{x}). \end{aligned}$$

The proof is similar for (b) except that $e^T Cx > e^T C\mathbf{x}$. □

Remark 2.1. In view of Propositions 2.1 and 2.4, $g(x)$ can be considered a measurement of the distance between x and $E(X, C)$. For if x is efficient, then $g(x) = 0$; and if $x \in X$ is not efficient, then $g(x) > 0$. Moreover, $g(x)$ is decreased by moving in a direction which dominates Cx (i.e. closer to the efficient set).

In the case where X is a polytope, this concept is enhanced by the continuity of g. When X is only assumed to be compact and convex, the only possible discontinuities of g occur at those boundary points of G which are not efficient.

3. Auxiliary Problems

Two auxiliary problems to (P) are now presented. The first is due to Benson. The second is motivated by Benon's result and the discussion in Section 2. This second problem will be used to provide the fundamental concepts for an algorithmic approach for solving (P).

Let $C^+ : R^n \to R^{k+1}$ defined by $C^+x = (c_1x, c_2x, \ldots, c_kx, \varphi(x))^T$.

$$(V^+) \quad \text{VMAX} C^+x \quad \text{subject to} \quad x \in X.$$

Problem (V^+) provides a necessary condition for a point to be an optimal solution for problem (P). This result was presented by Benson [5] in the special case when φ is a linear function. The proof given in [6] remains valid for an arbitrary function φ.

PROPOSITION 3.1. *If the point x^* is an optimal solution to (P), then $x^* \in E(X, C^+)$.*

Remark 3.1. Some benefits of this result are given in [5]. One drawback to Proposition 3.1 occurs when $E(X, C)$ is a subset of $E(X, C^+)$. In such cases the fact that $x^* \in E(X, C^+)$ provides no new information.

The problem (V^+) can be motivated by the desire to maximize the original k objectives and φ simultaneously. The next problem considered is motivated by the desire to simultaneously reward increases in φ and penalize being away from $E(X, C)$. In other words, this problem is designed to attempt to maximize φ and, at the same time, to minimize g, Benson's measurement function of Section 2.

$$(V\sim) \quad \text{VMAX}(\varphi(x), -g(x))^T \quad \text{subject to} \quad x \in X.$$

Let $L(x) = (\varphi(x), -g(x))^T$. Then the efficient set for $(V\sim)$ is $E(X, L)$.

Several properties of our auxiliary problem $(V\sim)$ are now presented. These properties will culminate with a condition that is both necessary and sufficient for a point to be an optimal solution for the main problem (P). The next result shows that $(V\sim)$ provides a necessary condition for locating optimal solutions of (P).

PROPOSITION 3.2. *If the point x^* is an optimal solution for (P), then $x^* \in E(X, L)$.*

Proof. Let x^* be an optimal solution for (P). Then $x^* \in E(X, C)$, and therefore, $g(x^*) = 0$. Suppose that x^* is not efficient for $(V\sim)$. Then there exists a point $x \in X$ such that $L(x)$ dominates $L(x^*)$. This implies that $-g(x) \geqslant -g(x^*) = 0$. By Proposition 2.1, 0 is the minimum of g on X. Therefore $g(x) = 0$ and so $x \in E(X, C)$. This implies that $\varphi(x) \leqslant \varphi(x^*)$. Since $g(x) = 0 = g(x^*)$ and $\varphi(x) \leqslant \varphi(x^*)$, it follows that $L(x) \leqslant L(x^*)$, which is a contradiction. □

The next two propositions offer a comparison between $(V\sim)$ and (V^+). The first establishes that $(V\sim)$ is at least as beneficial as (V^+); the second shows that the difficulty discussed in Remark 3.1 is avoided by $(V\sim)$.

PROPOSITION 3.3. $E(X, L) \subseteq E(X, C^+)$.

Proof. Suppose that $x \in E(X, L)$ and that x is not efficient for (V^+). Then there exists an $\mathbf{x} \in X$ such that $C^+\mathbf{x}$ dominates C^+x. It suffices to consider two cases: either $\varphi(\mathbf{x}) > \varphi(x)$ and $Cx \geqslant Cx$, or $\varphi(\mathbf{x}) = \varphi(x)$ and $C\mathbf{x}$ dominates Cx. In the second case, Proposition 2.4 (b) implies that $-g(\mathbf{x}) > -g(x)$. In either case, $L(\mathbf{x})$ is dominated by $L(x)$, which contradicts the assumption that $x \in E(X, L)$. □

PROPOSITION 3.4. *Let $\mathbf{x} \in X$. Then $\mathbf{x}$ is not efficient for $(V\sim)$ if*

$$\varphi(\mathbf{x}) < \sup \varphi(x) \quad \textit{subject to} \quad x \in E(X, C).$$

Proof. Assume that $\varphi(\mathbf{x}) < \sup \varphi(x)$ subject to $x \in E(X, C)$. Then there exists an $x \in E(X, C)$ such that $\varphi(\mathbf{x}) < \varphi(x)$. By Proposition 2.1, $-g(\mathbf{x}) \leqslant 0$ and $g(x) = 0$. Therefore $L(\mathbf{x})$ is dominated by $L(x)$, which implies $\mathbf{x}$ is not efficient for $(V\sim)$. □

Now a characterization of the optimal solutions of (P) can be given. This characterization will be utilized to motivate an algorithmic approach for solving (P).

THEOREM 3.1. *The point x^* is an optimal solution of (P) if and only if $x^* \in E(X, C) \cap E(X, L)$.*

Proof. The necessity is an immediate consequence of Proposition 3.2 and the definition of (P). Now assume that, $x^* \in E(X, C) \cap E(X, L)$. By Proposition 3.4, $\varphi(x^*) \geqslant \varphi(x)$ for all $x \in E(X, C)$. Since $x^* \in E(X, C)$, it follows that $\varphi(x^*) = \max \varphi(x)$, subject to $x \in E(X, C)$. That is, x^* is an optimal solution of (P). □

The full impact of the preceding results is best seen in the range space of $L(x)$, that is, $L(X) = \{(\varphi(x), -g(x))^T : x \in X\}$. Observe that the set of optimal solutions

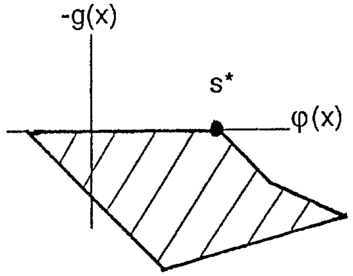

Fig. 1.

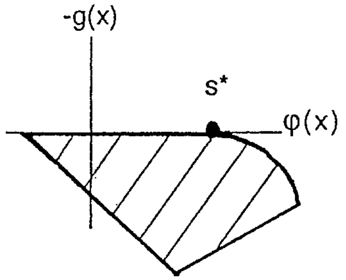

Fig. 2.

for (P) all map to the same point, namely the point $(\varphi(x^*), 0)^T$ where x^* is any optimal solution for (P). Let $s^* = (t^*, 0)$. To solve (P) it suffices to locate s^*. By using the theory developed earlier in this paper, the graph of $L(X)$ can be classified into two important cases, depicted in the Figures 1 and 2.

It should be noted that it is easy to show that $x^* \in E(X, C) \cap E(X, L)$ if and only if x^* is an optimal solution to the reverse convex program

$$\sup \varphi(x) \quad \text{subject to} \quad g(x) \leqslant 0, \;\; x \in X.$$

Such programs are becoming a solvable set of problems in global optimization. See [16,17].

Proposition 2.1 shows that the image of $E(X, C)$ under L is a subset of the horizontal axis in R^2 and that $L(x)$ lies below the horizontal axis whenever $x \notin E(X, C)$. The image of $E(X, L)$ forms the efficient set of $L(X)$. Because $L(X)$ is the range set, this image can be visualized as the set of points $L(X) \in L(X)$ such that $L(X) \cap (L(x) + R^{2+}) = \{L(x)\}$, where R^{2+} is the first quadrant. Hence the image of $E(X, L)$ is $E(L(X), I_2)$, which is denoted by $E(L(X))$.

The desired result is to locate s^* via $E(X, L)$. The above diagrams are not intended to imply that $E(L(X))$ is connected or that $L(X)$ is compact (although $L(X)$ is bounded). Indeed, if there is no optimal solution for (P) then $s^* \notin L(X)$. In general, $L(X)$ is not a convex set. Consequently, it is difficult to generate all of $E(L(X))$. However, the following parametric programming problem does provide a sufficient condition for locating points in $E(L(X))$. This program can be used

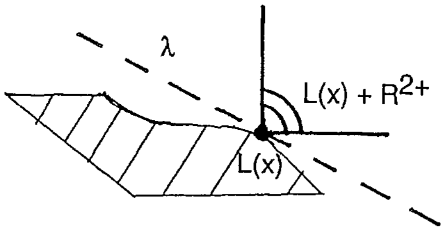

Fig. 3.

to either solve or provide an approximate solution for (P). For a positive scalar λ, define the problem

$$(P\lambda) \quad \max(1, \lambda)(\varphi(x), -g(x))^T \quad \text{subject to} \quad x \in X.$$

Remark 3.2. Using Proposition 2.3, an optimal solution of $(P\lambda)$ is guaranteed to exist if X is a polytope. Numerical techniques for solving $(P\lambda)$ will be discussed in Section 5.

PROPOSITION 3.5. *[18]. Let $\lambda > 0$ be given. If x is an optimal solution of $(P\lambda)$, then $x \in E(X, L)$.*

COROLLARY 3.1. *If x is an optimal solution for $(P\lambda)$ for some $\lambda > 0$ and $g(x) = 0$, then* **x** *is an optimal solution of (P).*

Proof. Assume that x is an optimal solution of $(P\lambda)$ for some $\lambda > 0$ and that $g(x) = 0$. By Proposition 3.5, $x \in E(X, L)$. By Proposition 2.1, $x \in E(X, C)$. Therefore, Theorem 3.1 implies that x is an optimal solution of (P). □

Remark 3.3. There is a meaningful geometric interpretation of Proposition 3.5. Let x be an optimal solution to $(P\lambda)$ for some $\lambda > 0$. Then there exists a supporting line for $L(X)$ which contains $L(x)$. Since $(1, \lambda)^T > 0$, this line separates $L(X)$ and $L(x) + R^{2+}$. See Figure 3. Therefore, $L(x) \in E(L(X))$.

Notice that there is some λ such that solving $(P\lambda)$ will locate s^* in Figure 1. That is to say, a line with a positive normal can be drawn which contains s^* but no other point of $L(X)$. Observe that when the normal is vertical, the supporting line to $L(X)$ contains all of the image of $E(X, C)$. This fails to discern between the points of $E(X, C)$. Hence the normal is required to be of the form $(1, \lambda)^T$ with $\lambda > 0$.

In the cases where $L(X)$ has a graph like Figure 2, $(P\lambda)$ cannot isolate s^* because the only supporting line to $L(X)$ containing s^* has a vertical normal. However, $(P\lambda)$ is still useful. The geometry suggests that if λ is increased, the normal becomes more vertical and the supporting line to $L(X)$ becomes more horizontal. Consequently, the point $L(x) \in L(X)$ which $(P\lambda)$ locates will be closer to s^*. This is proven in the next section. Hence, even if $L(X)$ has the appearance of Figure 2, an approximate solution to (P) can be generated.

In general, one cannot predict if the graph of $L(X)$ will look like Figure 1 or Figure 2. However, by solving $(P\lambda)$ for larger and larger λ, either (P) is solved, or a reasonable approximation of t^* is obtained. Corollary 3.1 can be used in this process to see if t^* has been located. Theorem 4.1, in the next section, provides both an upper and lower bound for t^*.

4. Main Results

In this section are presented three theorems relating the solutions of $(P\lambda)$ to the solution of (P). The first result provides an interval of approximation which is improved by increasing λ. The other two results give conditions under which $(P\lambda)$ can be used to find the exact solution to (P).

THEOREM 4.1. *Let $\mathbf{x}$ be an optimal solution of $(P\boldsymbol{\lambda})$. Let $w \in W(\mathbf{x})$ be such that $g(\mathbf{x}) = e^T C(w - \mathbf{x})$. Then $\varphi(w) \leqslant t^* \leqslant \varphi(\mathbf{x}) - \boldsymbol{\lambda} g(\mathbf{x})$. Moreover, if $\lambda > \boldsymbol{\lambda}$ and x is an optimal solution of $(P\lambda)$, then $t^* \leqslant \varphi(x) - \lambda g(x) \leqslant \varphi(\mathbf{x}) - \boldsymbol{\lambda} g(\mathbf{x})$.*

Proof. Suppose that $\mathbf{x}$ is an optimal solution of $(P\boldsymbol{\lambda})$ and that $w \in W(\mathbf{x})$ satisfies $g(\mathbf{x}) = e^T C(w - \mathbf{x})$. Then $w \in E(X, C)$ by Proposition 2.2. Therefore, $\varphi(w) \leqslant t^*$. By the definition of $(P\lambda)$, $\varphi(x) - \boldsymbol{\lambda} g(x) \leqslant \varphi(\mathbf{x}) - \boldsymbol{\lambda} g(\mathbf{x})$ for all $x \in X$. In particular, when $x \in E(X, C)$, then $g(x) = 0$. Therefore, $\varphi(x) \leqslant \varphi(\mathbf{x}) - \boldsymbol{\lambda} g(\mathbf{x})$ for all $x \in E(X, C)$, and consequentially $t^* \leqslant \varphi(\mathbf{x}) - \boldsymbol{\lambda} g(\mathbf{x})$.

Now suppose that $\lambda > \boldsymbol{\lambda}$ and that x is an optimal solution for $(P\lambda)$. Then, because $\mathbf{x}$ is optimal for $(P\boldsymbol{\lambda})$,

$$\varphi(\mathbf{x}) - \boldsymbol{\lambda} g(\mathbf{x}) \geqslant \varphi(x) - \boldsymbol{\lambda} g(x). \tag{4.1}$$

Since $g(x) \geqslant 0$ and $\lambda > \boldsymbol{\lambda}$, it follows that $\lambda g(x) \geqslant \boldsymbol{\lambda} g(x)$. Hence $-\boldsymbol{\lambda} g(x) \geqslant -\lambda g(x)$. Combining this with (4.1) yields $\varphi(\mathbf{x}) - \boldsymbol{\lambda} g(\mathbf{x}) \geqslant \varphi(x) - \boldsymbol{\lambda} g(x) \geqslant \varphi(x) - \lambda g(x)$. □

Remark 4.1. Notice that $(\varphi(x) - \lambda g(x), 0)$ is the point where the supporting line to $L(X)$ intersects the horizontal axis. Hence, as λ increases this intercept moves to the left.

COROLLARY 4.1. *Let $\lambda > \boldsymbol{\lambda} > 0$ and assume that x and $\mathbf{x}$ are optimal solutions to $(P\lambda)$ and $(P\boldsymbol{\lambda})$ respectively. If $\varphi(\mathbf{x}) - \boldsymbol{\lambda} g(\mathbf{x}) = t^*$, then $\varphi(x) = t^*$.*

Proof. By Theorem 4.1 it follows that $t^* \leqslant \varphi(x) - \lambda g(x) \leqslant \varphi(\mathbf{x}) - \boldsymbol{\lambda} g(\mathbf{x}) = t^*$. Hence $\varphi(x) - \lambda g(x) = t^*$, which implies $(1, \lambda)(\varphi(x), -g(x))^T = t^*$. Since $(1, \lambda)(t^*, 0)^T = t^*$, then the vector $(\varphi(x) - t^*, -g(x))$ is orthogonal to $(1, \lambda)$.

This implies that for some scalar $\beta \geqslant 0$, $(\varphi(x) - t^*, -g(x)) = \beta(\lambda, -1)$. If $g(x) > 0$ then $\beta > 0$. In this case,

$$\begin{aligned}(1,\lambda)(\varphi(x),-g(x))^T &= (1,\boldsymbol{\lambda})(\varphi(x)-t^*,-g(x))^T + t^* \\ &= (1,\lambda)\beta(\lambda,-1)^T + t^* \\ &= \beta(\lambda - \boldsymbol{\lambda}) + t^* \\ &> t^*.\end{aligned}$$

This contradicts the optimality of **x** for $(P\boldsymbol{\lambda})$. Therefore $g(x) = 0$, which implies $\varphi(x) = t^*$. □

COROLLARY 4.2. *Let g be continuous on X. Let $f(\lambda)$ be the optimal value for $(P\lambda)$. Then* $\lim_{\lambda\to\infty} f(\lambda) = t^*$.

Proof. Since g is continuous on X, then $f(\lambda)$ exists for all $\lambda > 0$. By Theorem 4.1, f is nonincreasing in λ. Therefore, it suffices to show that the sequence $f(n)$, where $n = 1, 2, 3, \ldots$, converges to t^*. According to Theorem 4.1, this sequence is decreasing and bounded below by t^*. Consequently, $f(n)$ converges to some real number $m \geqslant t^*$. By definition of $f(n)$, there exists a sequence x_n from X such that $f(n) = \varphi(x_n) - ng(x_n)$ for $n = 1, 2, 3, \ldots$ Since X is compact, there exists a convergent subsequence of x_n, say x_{n_k}, which converges to $x \in X$. By continuity, $\varphi(x_{n_k}) \to \varphi(x)$ and $g(x_{n_k}) \to g(x)$. Since $\varphi(x_{n_k}) - n_k g(x_{n_k})$ has limit m, it follows that $g(x) = 0$. Therefore, $x \in E(X, C)$. Since $\varphi(x_{n_k}) - n_k g(x_{n_k})$ decreases to m an $n_k g(x_{n_k}) \geqslant 0$, it follows that $\varphi(x_{n_k}) \geqslant m$. Consequently, $\varphi(x) \geqslant m \geqslant t^*$. Since $x \in E(X, C)$, then $\varphi(x) \leqslant t^*$. Therefore, $\varphi(x) = t^*$, which implies $\lim f(n) = m = t^*$. □

Remark 4.2. In many applications the set $\{x \in R^n : Cx = 0\}$ is multi-dimensional. Hence in Theorem 4.1, any $\mathbf{w} \in \{\mathbf{w} \in X : C\mathbf{w} = Cw\}$ will satisfy $g(\mathbf{x}) = e^T C(\mathbf{w} - x)$. Therefore, the left hand side of the approximation is difficult to control. However, increasing λ can only decrease the upper bound. Moreover, if **x** is an optimal solution of some $(P\boldsymbol{\lambda})$ and satisfies $g(\mathbf{x}) = 0$, then t^* has been found and **x** is an optimal solution to (P). In this case, solving $(P\lambda)$ for $\lambda > \boldsymbol{\lambda}$ will continue to yield optimal solutions of (P). Consequently, no λ is too large. Conditions that guarantee such a $\boldsymbol{\lambda}$ exists are presented next.

Let $\mathrm{co}L(X)$ denote the convex hull of $L(X)$ and denote the efficient set of $\mathrm{co}L(X)$ under the identity map by $E(coL(X))$. The I_2 will also be suppressed when denoting the efficient set of $\mathrm{co}L(X) - R^{2+}$ under the identity map.

LEMMA 4.1. *Let x^* be an optimal solution of (P). Then $s^* = L(x^*)$ is an efficient extreme point of the convex hull of $L(X)$.*

Proof. Assume that x^* is an optimal solution of (P). Suppose that $s^* \notin E(\mathrm{co}L(X))$. Then there is a point $u \in \mathrm{co}L(X)$ such that u dominates s^*. That is, there exists a finite convex combination of points in $L(X)$ which dominates s^*.

That is, there exists for $1 \leqslant j \leqslant m$, the points $x_j \in X$ and scalars α_j such that $\alpha_j \geqslant 0$, $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 1$ and $u = \alpha_1 L(x_1) + \alpha_2 L(x_2) + \cdots + \alpha_m L(x_m)$. Assume, without loss of generality, that $\alpha_j > 0$ for $1 \leqslant j \leqslant m$. Now consider $u = \alpha_1 L(x_1) + \cdots + \alpha_m L(x_m)$, which dominates $s^* = (t^*, 0)^T$. From the second component of this equation it follows that $-[\alpha_1 g(x_1) + \alpha_2 g(x_2) + \cdots + \alpha_m g(x_m)] \geqslant 0$. Because $\alpha_j > 0$ and $g(x_j) \geqslant 0$ for $1 \leqslant j \leqslant m$, it follows that $0 \geqslant -[\alpha_1 g(x_1) + \alpha_2 g(x_2) + \cdots + \alpha_m g(x_m)] \geqslant 0$. This implies that $\alpha_1 g(x_1) + \alpha_2 g(x_2) + \cdots + \alpha_m g(x_m) = 0$. Since $\alpha_j > 0$ and $g(x_j) \geqslant 0$ for $1 \leqslant j \leqslant m$, then $g(x_j) = 0$ for $1 \leqslant j \leqslant m$. Therefore, $x_j \in E(X, C)$ and consequently $\varphi(x_j) \leqslant t^*$ for $1 \leqslant j \leqslant m$. This fact and the assumption that $s^* \leqslant u$ imply that

$$\begin{aligned} t^* &\leqslant \alpha_1 \varphi(x_1) + \alpha_2 \varphi(x_2) + \cdots + \alpha_m \varphi(x_m) \\ &\leqslant (\alpha_1 + \alpha_2 + \cdots + \alpha_m) t^* = t^*. \end{aligned}$$

Since $\alpha_j > 0$ and $\varphi(x_j) \leqslant t^*$ for $1 \leqslant j \leqslant m$, it follows that $\varphi(x_j) = t^*$ for $1 \leqslant j \leqslant m$. This implies that $u = s^*$. Therefore $s^* \in E(\mathrm{co}L(X))$.

Now assume that s^* is not an extreme point of $\mathrm{co}L(X)$. Then there exists the points u, $v \in \mathrm{co}L(X)$ such that $u \neq s^*$, $v \neq s^*$ and $s^* = \alpha u + (1 - \alpha)v$ for some $0 < \alpha < 1$. Let $u = (u_1, u_2)^T$ and $v = (v_1, v_2)^T$. It was shown above that $u_2 \leqslant 0$ and $v_2 \leqslant 0$. Therefore, $0 = \alpha u_2 + (1 - \alpha)v_2 \leqslant 0$ with equality occurring if and only if $u_2 = v_2 = 0$ since $0 < \alpha < 1$. Combining this fact with the knowledge that $s^* \in E(\mathrm{co}L(X))$ gives $u_1 \leqslant t^*$ and $v_1 \leqslant t^*$. Therefore $t^* = \alpha u_1 + (1 - \alpha)v_1 \leqslant \alpha t^* + (1 - \alpha)t^* = t^*$. Once again, $0 < \alpha < 1$ implies that $u_1 = v_1 = t^*$. Therefore $u = v = s^*$. Hence s^* is an extreme point of $\mathrm{co}L(X)$. □

THEOREM 4.2. *Let X be a polytope. Suppose that φ is a convex function on X. Then there exists a $\boldsymbol{\lambda} > 0$ such that for all $\lambda > \boldsymbol{\lambda}$, if x^* is an optimal solution of $(P\lambda)$, then x^* is an optimal solution of (P).*

Proof. The assumption that X is a polytope guarantees that (P) has an optimal solution and that g is continuous. Hence $(P\lambda)$ has an optimal solution for all $\lambda > 0$. Moreover, L is continuous, which implies that $L(X)$ is compact. This in turn guarantees that $\mathrm{co}L(X)$ is compact [19, p. 158]. Consequently, the set $\mathrm{co}L(X) - R^{2+}$ is a closed convex set [19, p. 75]. Therefore, $\mathrm{co}L(X) - R^{2+}$ is the intersection of the closed half-spaces which contain it [19, p. 99].

We claim that $\mathrm{co}L(X) - R^{2+}$ is a polyhedron. The details of this are rather lengthy (see [14]). To see this note that by the above arguments it suffices to show that $\mathrm{co}L(X) - R^{2+}$ has finitely many distinct exposed faces [19, p. 162].

Let F denote such a face of $\mathrm{co}L(X) - R^{2+}$. Then for some scalars α_1 and α_2, F is the optimal solution set of $\max(\alpha_1, \alpha_2) \cdot v$ subject to $v \in \mathrm{co}L(X) - R^{2+}$. Since $-i$ and $-j$ are feasible directions from any $u \in \mathrm{co}L(X) - R^{2+}$, then $\alpha_1 \geqslant 0$ and $\alpha_2 \geqslant 0$. Therefore, $(\alpha_1, \alpha_2)^T \geqslant 0$, and it suffices to show that there are finitely many exposed faces corresponding to normals of the form $(1, \lambda)^T$, where $\lambda > 0$. Now, let $\lambda > 0$ and consider

$$\text{(4.2)} \qquad \max(1, \lambda) \cdot u \quad \text{subject to} \quad u \in \text{co}L(X).$$

By the compactness of $L(X)$, the maximum is achieved, say at u^*. Suppose there exists $v \in \text{co}L(X) - R^{2+}$ such that $(1, \lambda) \cdot v > (1, \lambda) \cdot u^*$. Then for some $(\beta_1, \beta_2)^T \in R^{2+}$ and some $u \in \text{co}L(X)$, $v = u - (\beta_1, \beta_2)^T$. Consequently,

$$(1, \lambda) \cdot u^* < (1, \lambda) \cdot v = (1, \lambda) \cdot u - \beta_1 - \beta_2\lambda \leqslant (1, \lambda) \cdot u.$$

This contradicts the optimality of u^* for (4.2). Since $u^* \in \text{co}L(X) - R^{2+}$, then $(1, \lambda) \cdot u^* = \max(1, \lambda) \cdot v$ subject to $v \in \text{co}L(X) - R^{2+}$. Therefore, the optimal solutions to (4.2) form a subset of F. By the compactness of $\text{co}L(X)$, the solutions to (4.2) contain an extreme point of $\text{co}L(X)$, which in turn is an extreme point of $L(X)$ [19, p. 65]. Therefore,

$$\begin{aligned}
&\max(1, \lambda) \cdot v \quad \text{subject to} \quad v \in \text{co}L(X) - R^{2+} \\
&\quad = \max(1, \lambda) \cdot u \quad \text{subject to} \quad u \in \text{co}L(X) \\
&\quad = \max(1, \lambda) \cdot s \quad \text{subject to} \quad s \in L(X) \\
&\quad = \max \varphi(x) - \lambda g(x) \quad \text{subject to} \quad x \in X.
\end{aligned}$$

Since g is concave, $\lambda > 0$ and φ is convex, it follows that $\varphi - \lambda g$ is convex [19, p. 33]. Consequently the value

$$\max \varphi(x) - \lambda g(x) \quad \text{subject to} \quad x \in X$$

is attained at an extreme point of X [19, p. 345]. The image of this extreme point lies on F. Hence each exposed face of $\text{co}L(X) - R^{2+}$ with a positive normal contains the image of an extreme point of X. Because $L(X) \subseteq R^2$, any point in $\text{co}L(X) - R^{2+}$ can lie on at most 3 exposed faces (a point, 2 lines). Hence the number of exposed faces of $\text{co}L(X) - R^{2+}$ with a positive normal is less than 3 times the number of vertices of X. Since X is a polytope, X has finitely many vertices. This establishes that $\text{co}L(X) - R^{2+}$ is a polyhedron.

Now let x^* be an optimal solution to (P). By Lemma 4.1, $s^* = (\varphi(x^*), 0)^T \in E(\text{co}L(X))$ and consequently $s^* \in E(\text{co}L(X) - R^{2+})$ [18]. Since $\text{co}L(X) - R^{2+}$ is a polyhedron, there exists a $\boldsymbol{\lambda} > 0$ such that

$$\begin{aligned}
(1, \lambda) \cdot s^* &= \max(1, \lambda) \cdot v \quad \text{subject to} \quad v \in \text{co}L(X) - R^{2+} \\
&= \max \varphi(x) - \boldsymbol{\lambda} g(x) \quad \text{subject to} \quad x \in X.
\end{aligned}$$

The proof is completed by applying Remark 4.2 and Corollary 4.1. □

Remark 4.3. Bolintineanu [11] has shown that under the assumptions of Theorem 4.2 problem (P) has an extreme point optimal solution. In particular, Bolintineanu showed that the minimum of a quasi-concave function over the efficient set of a multiple objective linear program with bounded feasible region occurs at an extreme point of the feasible region.

Remark 4.4. To apply Theorem 4.2, $(P\lambda)$ is solved for some large λ. If $g(x^*) = 0$, then an optimal solution to (P) has been found. If not, then λ is increased and the process repeated. If φ is not convex or if X is not a polytope, then the conclusion of Theorem 4.2 cannot be guaranteed. In particular, the algorithm need not be finite and the supremum in (P) need not be attained.

EXAMPLE 4.1. Consider the nonpolytope

$$S = \{s = (x, y, z)^T \in R^3 : s \geqslant 0, x^2 + y^2 + z^2 \leqslant 1\}.$$

Let $Cs = (x, y)^T$. Then

$$E(S, C) = \{s \in S : x^2 + y^2 = 1, z = 0\}.$$

Setting $\varphi(s) = z$ it is immediate that

$$0 = \max \varphi(s) \quad \text{subject to} \quad s \in E(S, C)$$

In this case

$$g(s) = \left\{ \begin{array}{ll} \sqrt{2} - x - y & \text{if } (x, y)^T \leqslant \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)^T \\ \sqrt{1 - x^2} - y & \text{if } x > \frac{\sqrt{2}}{2} \\ \sqrt{1 - y^2} - x & \text{if } y > \frac{\sqrt{2}}{2} \end{array} \right\}$$

Given $\lambda > 0$, the optimal solution to $(P\lambda)$ is given by $x = y = \lambda/\sqrt{1 + 2\lambda^2}$, $z = 1/\sqrt{1 + 2\lambda^2}$, which yields the positive optimal value of

$$\sqrt{1 + 2\lambda^2} - \sqrt{2\lambda^2} \text{ for } (P\lambda).$$

EXAMPLE 4.2. Let S be the square in R^2 determined by $\{s = (x, y)^T : -1 \leqslant x \leqslant 0, 0 \leqslant y \leqslant 1\}$ and $Cs = (x + y, x - y)^T$. Then $E(S, C) = \{s \in S : x = 0\}$ and $g(s) = -2x$. Let $\varphi(s) = -3x^{1/3}$, which is not convex on S. Then

$$0 = \max \varphi(s) \quad \text{subject to} \quad s \in E(S, C).$$

Given $\lambda > 1/2$, the optimal solutions to $(P\lambda)$ occur when $x = -(2\lambda)^{-3/2}$, $0 \leqslant y \leqslant 1$. These solutions yield the positive optimal value of $\sqrt{2/\lambda}$ for $(P\lambda)$. If $\lambda \leqslant 1/2$ then the optimal solutions to $(P\lambda)$ occur at $x = -1$ with the positive objective $3 - 2\lambda$.

The following theorem gives a condition for which the program $(P\lambda)$ only needs to be solved for one value of λ. The sup norm of a vector x is denoted by $||x||_\infty$.

LEMMA 4.2. *Let $Cw \geqslant Cx$. Then for any $\alpha \in R^k$,*

$$|\alpha^T C(w-x)| \leqslant \|\alpha\|_\infty e^T C(w-x).$$

Proof. Let $\alpha \in R^k$ and assume $Cw \geqslant Cx$. Let $\beta = Cw - Cx$. Note $\beta \geqslant 0$. Therefore

$$\begin{aligned}
|\alpha^T C(w-x)| &= |\alpha^T \beta| \\
&= |\alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_k\beta_k| \\
&\leqslant |\alpha_1\beta_1| + |\alpha_2\beta_2| + \ldots + |\alpha_k\beta_k| \\
&= |\alpha_1|\beta_1 + |\alpha_2|\beta_2 + \ldots + |\alpha_k|\beta_k \\
&\leqslant \max|\alpha_i|(\beta_1 + \beta_2 + \ldots + \beta_k) \text{ for } 1 \leqslant i \leqslant k \\
&= \|\alpha\|_\infty(\beta_1 + \beta_2 + \ldots + \beta_k) \\
&= \|\alpha\|_\infty e^T \beta \\
&= \|\alpha\|_\infty e^T C(w-x).
\end{aligned}$$

□

THEOREM 4.3. *Suppose that $\varphi(x) = \alpha^T Cx$ for some $\alpha \in R^k$ and let $\lambda > \|\alpha\|_\infty$. Then x^* is an optimal solution for (P) if and only if x^* is an optimal solution for $(P\lambda)$.*

Proof. Assume that x^* is an optimal solution for (P). Then $x^* \in E(X, C)$, which implies $g(x^*) = 0$. Therefore, $\varphi(x^*) - \lambda g(x^*) = \alpha^T Cx^*$. Let $x \in X$. If $\alpha^T Cx \leqslant \alpha^T Cx^*$, then since $\lambda > 0$ and $g(x) \geqslant 0$, it follows that

$$\varphi(x) - \lambda g(x) = \alpha^T Cx - \lambda g(x) \leqslant \alpha^T Cx \leqslant \alpha^T Cx^*.$$

If $\alpha^T Cx > \alpha^T Cx^*$, then $x \notin E(X, C)$. Let $w \in W(x)$ satisfy $g(x) = e^T C(w - x)$. Since $x \notin E(X, C)$, $g(x) > 0$. Since$\lambda > \|\alpha\|_\infty$ and $g(x) > 0$, it follows that $\|\alpha\|_\infty e^T C(w-x) < \lambda e^T C(w-x)$. Hence by Lemma 4.2, $\lambda e^T C(w-x) > |\alpha^T C(w-x)|$. Therefore

$$\begin{aligned}
\varphi(x) - \lambda g(x) &= \alpha^T Cx - \lambda e^T C(w-x) \\
&< \alpha^T Cx - |\alpha^T C(w-x)| \\
&\leqslant \alpha^T Cx - \alpha^T Cx + \alpha^T Cw \\
&= \alpha^T Cw.
\end{aligned}$$

By Proposition 2.2, w $\in E(X, C)$. Therefore $\varphi(x) - \lambda g(x) < \alpha^T Cw \leqslant \alpha^T Cx^*$. Hence $\alpha^T Cx^*$ is the maximal value of $(P\lambda)$ and x^* is an optimal solution for $(P\lambda)$.

Now assume that x^* is an optimal solution of $(P\lambda)$. By Proposition 3.5, $x^* \in E(X, L)$. Hence by Theorem 3.1, it suffices to show that $x^* \in E(X, C)$. To show this suppose that $x^* \notin E(X, C)$. Let $w \in W(x^*)$ satisfy $g(x^*) = e^T C(w - x^*)$. By Proposition 2.1, $g(x^*) > 0$. Since $\lambda > \|\alpha\|_\infty$ and $g(x^*) > 0$ it follows that $||\alpha||_\infty e^T C(w - x^*) < \lambda e^T C(w - x^*)$. Applying Lemma 4.2, it follows that $|\alpha^T C(w - x^*)| < \lambda e^T C(w - x^*)$. Hence

$$\begin{aligned} \varphi(x^*) - \lambda g(x^*) &= \alpha^T Cx^* - \lambda e^T C(w - x^*) \\ &< \alpha^T Cx^* - |\alpha^T C(w - x^*)| \\ &\leqslant \alpha^T Cx^* + \alpha^T C(w - x^*) = \alpha^T Cw. \end{aligned}$$

However, by Propositions 2.1 and 2.2, $g(w) = 0$. Hence $\varphi(w) - \lambda g(w) = \alpha^T Cw > \varphi(x^*) - \lambda g(x^*)$, which contradicts the optimality of x^* for $(P\lambda)$. Therefore $x^* \in E(X, C)$. □

Remark 4.5. If $\alpha \geqslant 0$ in Theorem 4.3, then $E(X, C) = E(X, C^+)$. Consequently, (P) can be reduced to the convex program

$$(Pc) \qquad \max \varphi(x) \quad \text{subject to} \quad x \in X.$$

If $\alpha > 0$, then the solutions to (P) and (Pc) coincide. If some component of α is zero, then the solutions of (Pc) need not be a subset of $E(X, C)$. However, an optimal solution to (P) can be found by using lexigraphical maximization where $\varphi(x)$ is maximized first. See [3] for a similar result.

Remark 4.6. To solve (P_i) for some $1 \leqslant i \leqslant k$, let $\varphi(x) = -c_i x$. The optimal solution set for (P_i) can be generated by solving the corresponding $(P\lambda)$ for any $\lambda > 1$.

Remark 4.7. Since X is allowed to be any compact convex set in Theorem 4.3, an optimal solution is not guaranteed to exist. (See [15].) If $\lambda \leqslant ||\alpha||_\infty$, then it is possible that an optimal solution of $(P\lambda)$ is not an optimal solution for (P), as the next example demonstrates.

EXAMPLE 4.3. Let X be the convex hull of the four points $(-1, 2)$, $(0, 2)$, $(1, 0)$ and $(-1, 0)$. Let $C = I_2$. Then $E(X, C)$ is the line segment from $(0, 2)$ to $(1, 0)$ and

$$g(x) = \left\{ \begin{array}{ll} 2 - x_1 - x_2 & \text{if } x_1 \leqslant 0 \\ 2 - 2x_1 - x_2 & \text{if } x_1 > 0 \end{array} \right\}$$

Consider the problem (P_1) which is to minimize x_1 subject to $x \in E(X, C)$. This minimum is 0 and is achieved at $(0, 2)$. The function corresponding to $(P\lambda)$ is

$$f(x_1, x_2, \lambda) = \left\{ \begin{array}{ll} (\lambda - 1)x_1 + (x_2 - 2)\lambda & \text{if } x_1 \leqslant 0 \\ (2\lambda - 1)x_1 + (x_2 - 2)\lambda & \text{if } x_1 > 0 \end{array} \right\}$$

which yields $f(0, 2, \lambda) = 0$ and $f(-1, 2, \lambda) = 1 - \lambda$. Consequently, a sufficient condition for $(P\lambda)$ to have only efficient solutions is that $\lambda > 1$.

Remark 4.8. If a decision maker is interested in optimizing a continuous function h over the set of efficient outcomes, the problem can be posed in domain space as

$$\sup h(Cx) \quad \text{subject to} \quad x \in E(X, C).$$

When h is linear, this problem falls into the category of Theorem 4.3 or Remark 4.5. If h is convex, then hC is convex [19, p. 38]. Consequently, Theorem 4.2 is applicable whenever X is a polytope. Numerical aspects of these cases are given in the following section.

5. Numerical Techniques

In this section are discussed some of the numerical options when X is a polytope and φ is a convex function on R^n. The numerical aspects of general nonlinear functions φ will be developed in later research.

Remark 5.1. Under the assumptions that X is a polytope and φ is a convex function, $(P\lambda)$ has a solution for each $\lambda > 0$ and Theorem 4.2 is applicable. For each λ, $(P\lambda)$ is a nonconvex programming problem involving the maximization of a continuous, convex function over a polytope. This problem has received considerable attention in the last twenty years and is a topic of ongoing research. Further discussion and an extensive bibliography can be found in [16]. An extensive survey article on concave minimization has been written by Benson [20].

Let $X = \{x \in R^n | Ax \leqslant b, x \geqslant 0\}$. Then by taking the dual of the linear program which defines g, it follows that

$$\begin{aligned} g(x) = -\max\ & e^T Cx + p^T Cx - u^T b \\ \text{subject to}\ \ & u^T A - p^T C \geqslant e^T C \\ & p, u \geqslant 0. \end{aligned}$$

This form of g has the advantage of a consistent constraint set instead of one which is dependent upon x. Using the above representation of g results in at the following program

$$(D\lambda) \qquad \max \varphi(x) + \lambda f(x) \quad \text{subject to} \quad Ax \leqslant b, x \geqslant 0$$

where

$$\begin{aligned} f(x) = \max\ & e^T Cx + p^T Cx - u^T b \\ \text{subject to}\ \ & u^T A - p^T C \geqslant e^T C \\ & p, u \geqslant 0. \end{aligned}$$

Remark 5.2. The function $f = -g$ is a continuous, piecewise linear, convex function on X. Therefore, solving $(D\lambda)$ in this form is once again a problem of maximizing a convex function over a polyhedron. It should be noted that the constraints involving the variables p and u need not form a bounded set.

The following theorem phrases the results of Section 4 in terms of the problem (D).

THEOREM 5.1. *Let φ be a convex function on R^n, and $X = \{x \in R^n | Ax \leqslant b, x \geqslant 0\}$. Then the following are valid.*

(i) If (x^*, p, u) is an optimal solution of $(D\lambda)$ and $e^T Cx^* + p^T Cx^* - u^T b = 0$, then x^* is an optimal solution for program (P).

(ii) There exists a $\boldsymbol{\lambda} > 0$ such that for all $\lambda > \boldsymbol{\lambda}$, if (x^*, p, u) is an optimal solution for $(D\lambda)$, then x^* is an optimal solution for (P).

(iii) In the case where $\varphi = \alpha^T Cx$ for some $\alpha \in R^k$ and $\lambda > \|\alpha\|_\infty$, if (x^*, p, u) is an optimal solution of $(D\lambda)$, then x^* is an optimal solution for program (P).

When $\varphi(x) = d^T x$ for some $d \in R^n$, then $(D\lambda)$ may be solved using the following program

$$(D\lambda\text{-linear}) \qquad \max d^T x + \lambda(e^T Cx + p^T Cx - u^T b)$$
$$\text{subject to} \quad Ax \leqslant b, \;\; u^T A - p^T C \geqslant e^T C, \;\; x, p, u \geqslant 0.$$

Remark 5.3. The preceding problem is known as a bilinear programming problem. This type of problem has also received much attention in recent years [16].

Acknowledgement

The authors would like to thank H. Benson (University of Florida, U.S.A.) and S. Bolintineanu (Perpignan University, France) for their helpful comments and suggestions during the writing of this work.

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